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Quaternionic Hilbert space and colour confinement: I

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Abstract. A formalism based on quaternionic Hilbert spaces is developed to describe the coloured hadron states. The definition of the multi-particle states consistent with the quaternionic structure is given. This definition implies the selection rules equivalent to the colour confinement. It is found that for the symmetry group $G = G_F \times SU(3)$ this theory is algebraically equivalent to the Fritzsch and Gell–Mann model.

1. Introduction

Recently the description of coloured quark states has been proposed by Günaydin and Gürsey (1973, 1974), Gürsey (1974, 1976), Günaydin (1973, 1976) (see also Gürsey *et al* (1976), Gürsey and Sikivie (1976), Bucella *et al* (1977)) in the context of octonionic quantum mechanics. Günaydin and Gürsey suggest that the octonionic scheme is a realisation of the Fritzsch and Gell–Mann (1973) proposal with natural algebraic confinement of colour. Their arguments were based on the statement that the Birkhoff–von Neumann propositional calculus cannot be realised in octonionic Hilbert space (OHS). However Günaydin *et al* (1978) have shown that in one-particle OHS the axioms of quantum mechanics remain unaffected. The analogous result was obtained by Rembieliński (1978) who has given a systematic description of multi-particle states. Moreover, Kosiński and Rembieliński (1978) have shown that theories based on the OHS have unacceptable features such as the unobservability of two-fermion states and mixing between observable and unobservable ones. Thus the applicability of octonionic theories to the description of the elementary particles is rather questionable.

However, because of some interesting formal properties of the OHS theories it is possible that a suitable choice of scalar algebra and Hilbert–space geometry can lead to a natural realisation of some fundamental selection rules appearing in particle physics.

In this paper it is shown that theories based on quaternionic Hilbert spaces (QHS) with complex geometry are free from the pathologies of octonionic theories. This holds because the selection rules for construction of the multi-particle states are less restrictive. In particular, if the theory possesses the symmetry group $G = G_F \times SU(3)$, then total algebraic colour confinement holds. The quark model based on this group is exactly equivalent to the Fritzsch and Gell–Mann model with natural quark confinement.

The study of quaternion quantum mechanics was undertaken by a number of authors (Finkelstein *et al* 1962, 1963, Jauch 1968, Emch 1963, 1972). However the QHS with quaternionic geometry was employed in these papers.

The plan of this paper is as follows: In § 2 a short review of the QHS formalism is given, providing readers with the notion of the QHS with complex geometry. It is shown that this QHS is isomorphic to the complex Hilbert space (CHS) with appropriate structure essentially determined by the self-representation of the unitary group U(2). The construction of the multi-particle states (§ 3) is based on this fact. In § 4 the problem of definition of the physical states is considered. It is found that the selection rules for construction of the multi-particle states can be interpreted as algebraic colour-confinement. Finally the equivalence with the Fritzsch and Gell-Mann model is proved for $G_F \times SU(3)_C$ invariant theory.

2. Quaternionic Hilbert space with complex geometry

In this section a short review of the QHS formalism is given. In particular the isomorphism between QHS and CHS is explained.

As is well known, there are three bilinear forms over the quaternionic algebra Q which define the norm

$$|A| = (a_\mu a_\mu)^{1/2} = (A_\alpha^\times A_\alpha)^{1/2}$$

with the property

$$|AB| = |A| \,.\, |B|:$$

(*a*) the quaternionic scalar product $\bar{A}B \in Q$,

$$\bar{A}B = e_0(a_\mu b_\mu) + \boldsymbol{e}(a_0\boldsymbol{b} - b_0\boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{b});$$

(*b*) the complex scalar product

$$\tfrac{1}{2}[\bar{A}B + \widetilde{(\bar{A}B)}] = A_\alpha^\times B_\alpha \in \mathscr{C};$$

(*c*) the real scalar product

$$\tfrac{1}{2}(\bar{A}B + \overline{(\bar{A}B)})] = a_\mu b_\mu \in \mathbb{R}.$$

Here the real quaternionic basis e_μ is used with the multiplication rules $e_i e_k = -e_0\delta_{ik} + \epsilon_{ikj}e_j$, $e_0 e_\mu = e_\mu e_0 = e_\mu$, $\mu = 0, 1, 2, 3$; $i, k, j = 1, 2, 3$. A real quaternion A can be represented as $A = e_\mu a_\mu$, $a_\mu \in \mathbb{R}$, or in complex form (sympletic decomposition) as $A = e_0 A_0 + e_1 A_1 \equiv e_\alpha A_\alpha$, $\alpha = 0, 1$, where $A_0 = e_0 a_0 + e_3 a_3$ and $A_1 = e_0 a_1 - e_3 a_2$ belong to the subfield $\mathscr{C}$ of Q spanned by e_0 and e_3 and $\mathscr{C}$ is isomorphic to the field of complex numbers. The conjugation operations are defined as follows:

$$\bar{A} \doteqdot e_0 a_0 - \boldsymbol{e}\boldsymbol{a} = e_0 A_0^\times - e_1 A_1 \qquad \text{(quaternionic conjugation)},$$

$$A^\times \doteqdot e_0 a_0 + e_1 a_1 - e_2 a_2 - e_3 a_3 = e_0 A_0^\times + e_1 A_1^\times \qquad \text{(complex conjugation)},$$

$$\tilde{A} \doteqdot e_0 a_0 - e_1 a_1 - e_2 a_2 + e_3 a_3 = e_0 A_0 - e_1 A_1.$$

In the following the complex scalar product (*b*) is denoted by $\langle A, B\rangle = A_\alpha^\times B_\alpha$. The definition (*b*) implies useful rules

$$\langle A, B\rangle = \langle B, A\rangle^\times = \langle B^\times, A^\times\rangle, \qquad \langle A, AB\rangle = \tfrac{1}{2}(B + \tilde{B})|A|^2,$$

$$\langle A, B + C\rangle = \langle A, B\rangle + \langle A, C\rangle, \qquad \langle A, B\alpha\rangle = \alpha\langle A, B\rangle \text{ if } \alpha \in \mathscr{C}$$

but in general $\langle A, \alpha B\rangle \neq \alpha\langle A, B\rangle$. Note that this complex bilinear form is invariant under transformations of the unitary group U(2).

Because the notion of the QHS with complex geometry (to the author's knowledge) was not used in the physical literature† it is introduced below by slight modification of the standard CHS postulates.

Postulate 1 (algebraic)

$\mathscr{H}$ is a linear vector space over the field of the quaternions, i.e.

(*a*) $\mathscr{H}$ is an additive abelian group

(*b*) the mapping $\mathscr{H} \times Q \to \mathscr{H}$ is defined which satisfies

(*i*) distributive laws

$$(f+g)A = fA + gA$$

$$f(A+B) = fA + fB$$

(*ii*) associativity for the quaternions

$$(fA)B = f(AB)$$

(*iii*) $fe_0 = f$.

Here $f, g \in \mathscr{H}$, $A, B \in Q$ and the right-handed multiplication convention is adopted.

Postulate 2 (geometric)

There exists a complex-valued scalar product (f, g) defined for all f, g in $\mathscr{H}$ such that

(*a*) $(f, g+h) = (f, g) + (f, h)$

(*b*) $(f, g)^\times = (g, f)$

(*c*) $(f, fA) = \frac{1}{2}(A + \tilde{A})|f|^2$ where $|f|^2 = (f, f) \geqslant 0$ and $|f| = 0$ is equivalent to $f = 0$

(*d*) $(f\alpha, g) = \alpha^\times (f, g)$ for $\alpha \in \mathscr{C}$

Postulate 3 (topological)

$\mathscr{H}$ is complete.

From the above postulates it follows that a vector $f \in \mathscr{H}$ can be represented in the sympletic form

$$f = e_\alpha f_\alpha, \qquad \alpha = 0, 1 \tag{1a}$$

or in the Dirac notation by the 'ket'

$$|f\rangle = e_\alpha |f_\alpha\rangle. \tag{1b}$$

Here $f_\alpha = \langle e_\alpha, f\rangle$ are $\mathscr{C}$-valued and the possible extra indices are omitted. The scalar

† In the physical context the QHS with quaternionic geometry was used. For relations between QHS's with different geometries see Rembieliuski, 'Notes on the structure of the octonionic and quaternionic Hilbert spaces' (in preparation).

product has the form

$$(f, g) = \sum_{\alpha,\beta} (e_\alpha f_\alpha, e_\beta g_\beta) = \sum_{\alpha,\beta} \langle e_\alpha, e_\beta\rangle (f_\alpha, g_\beta)$$

$$= \sum_{\alpha} (f_\alpha, g_\alpha) = \sum_{\alpha} \langle f_\alpha \mid g_\alpha\rangle \equiv \langle f \mid g\rangle \tag{2}$$

because $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$ and the 'bra' vectors are defined by $\langle f| \doteq |\bar{f}\rangle^{\mathrm{T}} = \langle f_\alpha|\bar{e}_\alpha$, $\langle f_\alpha| \doteq |f_\alpha\rangle^{+} = |f_\alpha^{\times}\rangle^{\mathrm{T}}$. The symbols (f_α, g_α) and $\langle f_\alpha \mid g_\alpha\rangle$ denotes the ordinary (complex) Hilbert-space scalar product.

The linear manifold is defined as a closed subset of $\mathscr{H}$ containing together with the vectors f and g all their linear combinations with complex coefficients. The linear operator $\mathscr{L}$ is a $\mathscr{C}$-linear mapping of the manifold $\mathscr{M} \subset \mathscr{H}$ into $\mathscr{H}$. In an orthonormal basis $\{|a\rangle\}$ it can be represented in the form

$$\mathscr{L} = \sum_{a,b} |a\rangle \mathscr{L}_{ab} \langle b|. \tag{3}$$

Here $\mathscr{L}_{ab} = \langle a|\mathscr{L}|b\rangle \in \mathscr{C}$ and note that in general $|a\rangle\mathscr{L}_{ab}\langle b| \neq \mathscr{L}_{ab}|a\rangle\langle b|$ or $|a\rangle\langle b|\mathscr{L}_{ab}$. However, the composition law for the linear operators $\mathscr{N}$ and $\mathscr{L}$ has the standard form

$$(\mathscr{N}\mathscr{L})_{ab} = \sum_{c} \mathscr{N}_{ac}\mathscr{L}_{cb}. \tag{4}$$

The Hermitian and unitary operators are defined as usual. They can be represented by Hermitian and unitary complex matrices respectively. The projectors on the manifolds are Hermitian but in general do not commute with multiplication by quaternions. This holds because linear manifolds are not generally closed under the multiplication operation.

It is easy to see that if we restrict ourselves to multiplication by complex scalars then postulates 1–3 reduce to the ordinary (CHS) ones. This fact and the definitions of the linear manifold and linear operator implies that the QHS is geometrically isomorphic to the CHS with appropriate structure: every vector $|f\rangle = e_\alpha|f_\alpha\rangle \in$ QHS is associated with a two-component complex vector $f \equiv \binom{f_0}{f_1} \in$ CHS. To every linear operator $\mathscr{L}$ acting in QHS there corresponds via relations (3) and (4) a linear operator in CHS. The scalar product in this CHS is obviously given by

$$\langle f \mid g\rangle \equiv (f, g) = \sum_{\alpha=0}^{1} (f_\alpha, g_\alpha).$$

The e_0 and e_3 are represented by 1 and $\mathrm{i} = (-1)^{1/2}$ respectively.

It is not difficult to prove that the above-mentioned CHS can also be equipped with the algebraic structure of the QHS. To do this it is sufficient to define the operations E_μ, $\mu = 0, 1, 2, 3$ which implement the multiplication of vectors by quaternionic units, and to determine the conjugation operations. This follows from the associativity of the quaternion algebra and postulate 1. From the form of the vector $|f\rangle$ (equation 1) it follows that

$$\begin{aligned}
E_0|f\rangle &\doteq |f\rangle e_0 = |f\rangle \\
E_1|f\rangle &\doteq |f\rangle e_1 = e_0(-|f_1\rangle)^{\times} + e_1(|f_0\rangle)^{\times} \\
E_2|f\rangle &\doteq |f\rangle e_2 = |f\rangle(e_3 e_1) = (|f\rangle e_3)e_1 = E_1 E_3|f\rangle \\
E_3|f\rangle &\doteq |f\rangle e_3 = e_0(|f_0\rangle e_3) + e_1(|f_1\rangle e_3)
\end{aligned} \tag{5}$$

and

$$\begin{aligned} C|f\rangle &\doteq (|f\rangle)^\times = e_0(|f_0\rangle)^\times + e_1(|f_1\rangle)^\times \\ D|f\rangle &\doteq |\tilde{f}\rangle = e_0|f_0\rangle + e_1(-|f_1\rangle) \\ F|f\rangle &\doteq e_0(|f_1\rangle e_3) + e_1(-|f_0\rangle e_3). \end{aligned} \tag{6}$$

Note that the quaternionic group with elements defined by the relations $E_{\mu_1 \ldots \mu_n}|f\rangle \doteq |f\rangle e_{\mu_n} \ldots e_{\mu_1}$ contains only eight operators $\pm E_\mu$ and that $E_1 = E_3FC$, $E_2 = FC$.

The geometrical correspondence between QHS and CHS implies via equations (5) and (6) the following representation for E_μ and C, D, F:

$$\begin{aligned} &E_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0, \qquad E_1 \equiv \mathrm{i}FC \\ &E_2 \equiv FC, \qquad E_3 \equiv \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} = \mathrm{i}\sigma_0, \\ &C\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0^\times \\ f_1^\times \end{pmatrix}, \qquad D \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \\ &F \equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = -\sigma_2. \end{aligned} \tag{7}$$

Thus the QHS is isomorphic geometrically and algebraically to the CHS defined above. Note that the whole quaternionic structure is essentially generated by E_0, E_3, C, D and F.

Now let us explain role of the group U(2) in the structure of QHS. As is well known the quaternion algebra admits the SO(3) as group of automorphisms. On the other hand U(2) is the invariance group of the scalar product in QHS. The algebraic and geometric structure of the QHS is unaffected under the action of the intersection of these groups $SO(3) \cap U(2) \sim U(1)$. A vector f transforms under this common subgroup U(1) as follows:

$$\begin{pmatrix} f_0' \\ f_1' \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{e}^{\mathrm{i}\phi} \end{array}\right) \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}. \tag{8}$$

The following fact is of great importance: the algebraic closure (under matrix multiplication) of the unitary matrices representing the operators $\pm E_0$, $\pm E_3$, $\pm E_1 C = \pm \mathrm{i}F$, $\pm E_2 C = \pm F$ (see equations 7) and the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{smallmatrix}\right)$ forms the self-representation **2** of the group U(2). Thus the geometrical 'gauge' group U(2) also determines the algebraic structure of the QHS.

Concluding, the whole structure of the QHS is essentially determined by the self-representation $\bar{\mathbf{2}}$ of U(2) (and its adjoint $\bar{\mathbf{2}}$ because $C : \mathbf{2} \to \bar{\mathbf{2}}$).

3. Tensor product of the quaternionic Hilbert spaces with complex geometry

The results of the preceding section allow us to define in a consistent way the 'tensor' product of the QHS. It is reasonable to demand that the resulting Hilbert space has a similar structure, i.e. it carries simultaneously irreducible representations of the group U(2) and the quaternionic group with a common subgroup containing the elements $\pm E_0$, $\pm E_3$, $\pm E_1 C = \pm \mathrm{i}F$ and $\pm E_2 C = \pm F$. More precisely, this indicates that the product space is a direct sum of the irreducible (with respect to both groups) subspaces. Because the quaternionic group has only two- and one-dimensional irreducible representations†, the above condition is satisfied only for one-dimensional (scalar or phase) and two-dimensional ($\mathbf{2}$ and $\bar{\mathbf{2}}$) representations of the group U(2). Thus if theory based on QHS formalism possesses a symmetry group G then

(*a*) G must necessarily contain U(2) as subgroup
(*b*) the only admissible representations D of G fulfil the condition

$$D(G) \downarrow \mathrm{U}(2) = (\oplus \mathbf{1}) \oplus (\oplus \mathbf{2}) \oplus (\oplus \bar{\mathbf{2}}) \tag{9}$$

i.e. their U(2) content is a direct sum of the representations $\mathbf{1}$ (scalar or phase), $\mathbf{2}$ and $\bar{\mathbf{2}}$. Consequently the consistent definition of the tensor product is given by

$$\mathscr{H}_1 \times \mathscr{H}_2 \times \ldots \times \mathscr{H}_n \doteqdot \Pi(\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_n) \tag{10}$$

Here the $\mathscr{H}_k$ are the carrier spaces of the admissible representations of G. The operator Π projects the ordinary tensor product ($\otimes$) of $\mathscr{H}_k$ on the whole subspace of the admissible representations of the group G. Note that this operation is almost analogous to the symmetrisation or antisymmetrisation of the multi-particle boson or fermion states respectively. However, there is very important difference because the quaternionic tensor product of some number of one-particle QHS cannot be obtained starting from one copy and multiplying successively by others. For example if G = U(2) then

$$\Pi(\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}) = \mathbf{2} \oplus \mathbf{2}$$

whereas

$$\Pi(\mathbf{2} \otimes (\mathbf{2} \oplus \mathbf{2})) = \mathbf{2}$$

and consequently

$$\mathscr{H}^2 \times \mathscr{H}^2 \times \mathscr{H}^2 = \mathscr{H}'^2 \otimes \mathscr{H}''^2 \neq \mathscr{H}^2 \times (\mathscr{H}^2 \times \mathscr{H}^2) = \mathscr{H}'''^2.$$

The multi-particle states can be generated from the vacuum by action of the quaternionic product of the field operators. This product is defined by the formula

$$\phi \times \phi \times \ldots \times \phi \doteqdot \Pi(\phi \otimes \phi \otimes \ldots \otimes \phi) \tag{11}$$

consistent with the tensor product definition (10).

4. Quaternionic Hilbert space and colour confinement

This section is devoted to a discussion of the following questions:

1. The definition of the physical states in QHS.
2. The interpretation of the structure group U(2) (denoted below by $\mathrm{U}(2)_c$).
3. The interpretation of the selection rules for admissible multi-particle states.
4. The classification problem of the admissible representations of the semi-simple compact Lie groups.

† For one dimensional (abelian) representations of the quaternionic group $\pm E_\mu \to 1$ or $\pm E_0$, $\pm E_3 \to 1$ and $\pm E_1$, $\pm E_2 \to C$ or $\pm E_0 \to 1$, $\pm E_3 \to -1$, $\pm E_1 \to C$, $\pm E_2 \to -C$. For a more exhaustive discussion of the product states see Rembieliński, 'Algebraical confinement of coloured states' (in preparation).

4.1. The physical (that is observable) states can be identified with singlets of the $SU(2)_c \subset U(2)_c$ *only.*

In fact, if the one-particle state f_α belongs to a doublet of $U(2)_c$ then the two-particle state $\phi_{\alpha\beta}$ belongs to the admissible representation $\Pi(\mathbf{2}\otimes\mathbf{2}) = \Pi(\mathbf{3}\oplus\mathbf{1}) = \mathbf{1}$, i.e. it forms a singlet of $SU(2)_c$. But this scalar is antisymmetrical, i.e. $\phi_{\alpha\beta} \sim f_\alpha(1)f_\beta(2) - f_\beta(1)f_\alpha(2)$. Thus two-particle states $\phi_{\alpha\alpha}$ do not in theory appear, because $\phi_{\alpha\alpha} = 0$. This feature is inadmissible for observable states. Consequently the doublets of $U(2)_c$ cannot be associated with observable states.

4.2.

The unobservability of doublets of the group $U(2)_c$ suggest that *the* $SU(2)_c$ *degrees of freedom should be identified with the colour.* Consequently the structure group $SU(2)_c$ should be a subgroup of the colour group.

4.3.

The definition (10) of the quaternionic tensor product implies strong selection rules on the acceptable multi-particle states. The multi-particle states belonging to the inadmissible representations of the symmetry group G do not in theory appear. Because the (admissible) doublets of $SU(2)_c$ are also unobservable, the *total algebraic colour confinement for the* $SU(2)_c$ *group holds.* It is to be expected that this mechanism works also for larger colour groups containing $SU(2)_c$. This problem is discussed in the following paper (Rembieliński 1979a) (see also below).

In conclusion the natural selection rules appearing in the QHS theories produce algebraic confinement of colour.

4.4.

The classification problem of the admissible representations of the (classical) semi-simple compact Lie groups is solved in the following paper (Rembieliński 1980). The results obtained strongly favour SU(3) as the colour group. It is shown that only the groups $SU(3n)$ where n is odd can be eventually identified with the colour group†. Furthermore, the only admissible representations of $SU(3n)$ are one dimensional (singlet), and $\binom{3n}{n}$ dimensional given by the following Young table (and its conjugate) (figure A). Note that only for SU(3) is it the self-representation. The $SU(3n)$ degrees of freedom are confined in all cases.

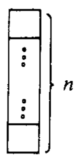

Figure A.

Let us consider a theory based on the symmetry group $G = G_F \times SU(3)_c$ (i.e. $G_{colour} = SU(3)$) where G_F is the flavour group and $SU(3)_c \supset U(2)_c$. The admissible

† The exceptional groups were not considered in this paper.

representations of G have the form $(D_F, \mathbf{1})$, $(D_F, \mathbf{3})$ and $(D_F, \bar{\mathbf{3}})$ where D_F denotes an arbitrary representation of G_F while $\mathbf{1}$, $\mathbf{3}$ and $\bar{\mathbf{3}}$ are the singlet, triplet and anti-triplet of $SU(3)_c$ respectively. As follows from the above discussion the $U(2)_c$ doublets are confined. Moreover the $SU(3)_c$ triplets (antitriplets) are also confined. In fact $\Pi(\mathbf{3}\otimes\mathbf{3}) = \Pi(\bar{\mathbf{3}} \oplus \mathbf{6}) = \bar{\mathbf{3}}$ ($\mathbf{6}$ is inadmissible) and anti-triplet $\bar{\mathbf{3}}$ is antisymmetric, i.e. the two-particle states do not exist in this case. Consequently the observable particles can be associated only with the singlets of the $SU(3)_c$. Thus the algebraic confinement of the $SU(3)_c$ colour holds, i.e. only the multiplets $(D_F, \mathbf{1})$ are observable. In particular the quarks associated with $SU(3)_c$ triplet are confined. So this theory is algebraically equivalent to the Fritzsch and Gell-Mann (1973) one.

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